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Research Article

Proof of One Optimal Inequality for Generalized Logarithmic, Arithmetic, and Geometric Means

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Two open problems were posed in the work of Long and Chu (2010). In this paper, we give the solutions of these problems.

1. Introduction

The arithmetic $A(a,b)$ and geometric $G(a,b)$ means of two positive numbers a and b are defined by $A(a,b) = (a+b)/2$, $G(a,b) = \sqrt{ab}$, respectively. If p is a real number, then the generalized logarithmic mean $L_p(a,b)$ with parameter p of two positive numbers a, b is defined by

$$L_p(a,b) = \begin{cases} a, & a = b, \\ \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p \neq 0,\ p \neq -1,\ a \neq b, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0,\ a \neq b, \\ \dfrac{b-a}{\ln b - \ln a}, & p = -1,\ a \neq b. \end{cases} \tag{1.1}$$

In the paper [1], Long and Chu propose the two following open problems:

Open Problem 1. What is the least value p such that the inequality

$$\alpha A(a,b) + (1-\alpha)G(a,b) < L_p(a,b) \tag{1.2}$$

holds for $\alpha \in (0,1/2)$ and all $a,b > 0$ with $a \neq b$?

Open Problem 2. What is the greatest value q such that the inequality

$$\alpha A(a,b) + (1-\alpha)G(a,b) > L_q(a,b) \tag{1.3}$$

holds for $\alpha \in (1/2,1)$ and all $a,b > 0$ with $a \neq b$?

For information on the history, background, properties, and applications of inequalities for generalized logarithmic, arithmetic, and geometric means, please refer to [1–19] and related references there in.

The aim of this article is to prove the following Theorem 2.1.

2. Main Result

Theorem 2.1. *Let $\alpha \in (0,1/2) \cup (1/2,1)$, $a \neq b$, $a > 0$, $b > 0$. Let $p(\alpha)$ be a solution of*

$$\frac{1}{p}\ln(1+p) + \ln\left(\frac{\alpha}{2}\right) = 0 \quad \text{in } (-1,1). \tag{2.1}$$

Then,

$$\text{if } \alpha \in \left(0,\frac{1}{2}\right), \text{ then } \alpha A(a,b) + (1-\alpha)G(a,b) < L_p(a,b) \text{ for } \ p \geq p(\alpha) \tag{2.2}$$

and $p(\alpha)$ is the best constant,

$$\text{if } \alpha \in \left(\frac{1}{2},1\right), \text{ then } \alpha A(a,b) + (1-\alpha)G(a,b) > L_p(a,b) \text{ for } p \leq p(\alpha) \tag{2.3}$$

and $p(\alpha)$ is the best constant.

3. Proof of Theorem 2.1

Because $L_p(a,b)$ is increasing with respect to $p \in \mathbb{R}$ for fixed a and b, it suffices to prove that for any $\alpha \in (0,1/2)$ (resp., $\alpha \in (1/2,1)$) there exists $p(\alpha)$ such that $\alpha A(a,b) + (1-\alpha)G(a,b) < L_{p(\alpha)}(a,b)$ (resp., $\alpha A(a,b) + (1-\alpha)G(a,b) > L_{p(\alpha)}(a,b)$), and $p(\alpha)$ is the best constant.

Without loss of generality, we assume that $a > b > 0$. Let $p \neq 0$, $p \neq -1$. Equations (2.2), (2.3) are equivalent to

$$\alpha\left(\frac{a+b}{2}\right) + (1-\alpha)\sqrt{ab} \lesseqgtr \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{1/p}. \tag{3.1}$$

On putting $t = \sqrt{b/a}$, we obtain (3.1) is equivalent to

$$\frac{1}{p}\ln\left(\frac{1-t^{2p+2}}{(p+1)(1-t^2)}\right) - \ln\left(\frac{\alpha}{2}\left(1+t^2\right) + (1-\alpha)t\right) \gtrless 0, \quad t \in (0,1). \tag{3.2}$$

Introduce the function $H : (0,1) \times (0,1) \times (-1,1) \to \mathbb{R}$ by

$$\begin{gathered} H(t,\alpha,p) = \frac{1}{p}\ln\left(\frac{1-t^{2p+2}}{(p+1)(1-t^2)}\right) - \ln\left(\frac{\alpha}{2}\left(1+t^2\right) + (1-\alpha)t\right), \quad p \neq 0, \\ H(t,\alpha,0) = \lim_{p\to 0} H(t,\alpha,p). \end{gathered} \tag{3.3}$$

Simple computations yield for $p \neq 0$

$$\begin{gathered} \frac{\partial H(t,\alpha,p)}{\partial t} = \frac{2}{p}\left(\frac{pt^{2p+3}-(p+1)t^{2p+1}+t}{(1-t^2)(1-t^{2p+2})}\right) - 2\left(\frac{\alpha t + 1 - \alpha}{\alpha(1+t^2)+2(1-\alpha)t}\right), \\ \frac{\partial H(t,\alpha,0)}{\partial t} = \lim_{p\to 0}\frac{\partial H(t,\alpha,p)}{\partial t}. \end{gathered} \tag{3.4}$$

Let $\alpha \in (0,1/2) \cup (1/2,1)$ and $p(\alpha)$ the unique solution to

$$\frac{1}{p}\ln(1+p) + \ln\left(\frac{\alpha}{2}\right) = 0. \tag{3.5}$$

To see that $p(\alpha)$ is optimal in both cases (2.2), (2.3), note that $\lim_{t\to 0^+} H(t,\alpha,p(\alpha)) = 0$. Thus, if the constant is decreased (resp., increased), then the desired bound for H would not hold for small t. This follows from the fact that for a fixed α, the function

$$H(0+,\alpha,p) = -\left(\frac{1}{p}\right)\ln(p+1) - \ln\left(\frac{\alpha}{2}\right) \tag{3.6}$$

is nondecreasing.

From now on, let $p = p(\alpha)$ for $\alpha \in (0, 1/2) \cup (1/2, 1)$. To show the estimates for this p, we start from observing that $H(0+, \alpha, p) = H(1-, \alpha, p) = 0$. Furthermore, one easily checks that

$$\begin{aligned} H_t'(0+, \alpha, p) &= \infty \quad \text{for } \alpha < \frac{1}{2}, \\ H_t'(0+, \alpha, p) &= \frac{2(\alpha - 1)}{\alpha} < 0 \quad \text{for } \alpha > \frac{1}{2}. \end{aligned} \tag{3.7}$$

Thus, it suffices to verify that $H_t'(\cdot, \alpha, p)$ has exactly one zero inside the interval $(0, 1)$. It follows from the mean value theorem. After some computations, this is equivalent to saying that the function R given by

$$\begin{aligned} R(t, \alpha, p) &= \ln\left(\frac{\alpha(p+1)t^3 + (1-\alpha)(p+2)t^2 + \alpha(1-p)t - p(1-\alpha)}{-p(1-\alpha)t^3 + \alpha(1-p)t^2 + (1-\alpha)(p+2)t + \alpha(p+1)}\right) \\ &\quad - (2p+1)\ln t = \ln\frac{s_1(t)}{s_2(t)} - (2p+1)\ln t \end{aligned} \tag{3.8}$$

has exactly one root in $(0, 1)$. Here, the expression under the logarithm may be nonpositive, so we define R on a maximal interval, contained in $(0, 1)$. It is easy to see that this interval must be of the form $(t_0, 1)$, for some $t_0 \in \langle 0, 1)$. This follows from the fact that s_2 is strictly positive on $\langle 0, 1\rangle$ and s_1 is strictly increasing on this interval.

Since $R(1-) = 0$ and $R(t_0+) = \pm\infty$, we will be done if we show that R' has exactly one root in $(0, 1)$. After some computations, we obtain that the equation $R'(t) = 0$ is equivalent to

$$g(t) = \alpha(1-\alpha)(2p+1)\left(1+t^2\right) + 2\left(p\left(2\alpha^2 - 2\alpha + 1\right) + \alpha^2 - 4\alpha + 2\right)t = 0. \tag{3.9}$$

Because g is a quadratic polynomial in the variable t, all that remains is to show that

$$g(0)g(1) = \alpha(1-\alpha)(2p+1)(p - 3\alpha + 2) < 0 \tag{3.10}$$

or, in virtue of the definition of $p = p(\alpha)$,

$$(2p+1)\left(p + 2 - \frac{6}{(p+1)^{1/p}}\right) < 0. \tag{3.11}$$

This can be easily established by some elementary calculations. It completes the proof.

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References

[1] B.-Y. Long and Y.-M. Chu, "Optimal inequalities for generalized logarithmic, arithmetic, and geometric means," *Journal of Inequalities and Applications*, vol. 2010, Article ID 806825, 10 pages, 2010.

[2] H. Alzer, "Ungleichungen für Mittelwerte," *Archiv der Mathematik*, vol. 47, no. 5, pp. 422–426, 1986.

[3] H. Alzer and S.-L. Qiu, "Inequalities for means in two variables," *Archiv der Mathematik*, vol. 80, no. 2, pp. 201–215, 2003.

[4] F. Burk, "The geometric, logarithmic, and arithmetic mean inequality," *The American Mathematical Monthly*, vol. 94, no. 6, pp. 527–528, 1987.

[5] W. Janous, "A note on generalized Heronian means," *Mathematical Inequalities & Applications*, vol. 4, no. 3, pp. 369–375, 2001.

[6] E. B. Leach and M. C. Sholander, "Extended mean values. II," *Journal of Mathematical Analysis and Applications*, vol. 92, no. 1, pp. 207–223, 1983.

[7] J. Sándor, "On certain inequalities for means," *Journal of Mathematical Analysis and Applications*, vol. 189, no. 2, pp. 602–606, 1995.

[8] J. Sándor, "On certain inequalities for means. II," *Journal of Mathematical Analysis and Applications*, vol. 199, no. 2, pp. 629–635, 1996.

[9] J. Sándor, "On certain inequalities for means. III," *Archiv der Mathematik*, vol. 76, no. 1, pp. 34–40, 2001.

[10] M.-Y. Shi, Y.-M. Chu, and Y.-P. Jiang, "Optimal inequalities among various means of two arguments," *Abstract and Applied Analysis*, vol. 2009, Article ID 694394, 10 pages, 2009.

[11] B. C. Carlson, "The logarithmic mean," *The American Mathematical Monthly*, vol. 79, pp. 615–618, 1972.

[12] J. Sándor, "On the identric and logarithmic means," *Aequationes Mathematicae*, vol. 40, no. 2-3, pp. 261–270, 1990.

[13] J. Sándor, "A note on some inequalities for means," *Archiv der Mathematik*, vol. 56, no. 5, pp. 471–473, 1991.

[14] T. P. Lin, "The power mean and the logarithmic mean," *The American Mathematical Monthly*, vol. 81, pp. 879–883, 1974.

[15] A. O. Pittenger, "Inequalities between arithmetic and logarithmic means," *Univerzitet u Beogradu. Publikacije Elektrotehničkog Fakulteta*, no. 678–715, pp. 15–18, 1980.

[16] C. O. Imoru, "The power mean and the logarithmic mean," *International Journal of Mathematics and Mathematical Sciences*, vol. 5, no. 2, pp. 337–343, 1982.

[17] C.-P. Chen, "The monotonicity of the ratio between generalized logarithmic means," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 1, pp. 86–89, 2008.

[18] X. Li, C.-P. Chen, and F. Qi, "Monotonicity result for generalized logarithmic means," *Tamkang Journal of Mathematics*, vol. 38, no. 2, pp. 177–181, 2007.

[19] F. Qi, S.-X. Chen, and C.-P. Chen, "Monotonicity of ratio between the generalized logarithmic means," *Mathematical Inequalities & Applications*, vol. 10, no. 3, pp. 559–564, 2007.